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the coefficients in each column on the left being in arithmetic progression from the principal diagonal downwards, and the differences of the progressions in adjacent columns forming the progression 3, 9, 15,  $\dots$ . Let us write the determinant solution for  $x_n$  in the form  $x_n = d_n'/d_n$ .

Then

$$d_n = \begin{vmatrix} 2 & 3 & 3 & \dots & 3 \\ 5 & 11 & 12 & \dots & 12 \\ 8 & 20 & 26 & \dots & 27 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

which becomes, if we subtract adjacent rows, leaving the first unchanged.

$$\begin{vmatrix} 2 & 3 & 3 & 3 & \dots & 3 \\ 3 & 8 & 9 & 9 & \dots & 9 \\ 3 & 9 & 14 & 15 & \dots & 15 \\ 3 & 9 & 15 & 20 & \dots & 21 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

and on repeating the process

$$\begin{vmatrix} 2 & 3 & 3 & 3 & \dots & 3 \\ 1 & 5 & 6 & 6 & \dots & 6 \\ 0 & 1 & 5 & 6 & \dots & 6 \\ 0 & 0 & 1 & 5 & \dots & 6 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

If now we perform the same operation once with columns, we get

$$d_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & 4 \end{vmatrix}.$$

Let  $d_{n-1}$ ,  $d_{n-2}$  be corresponding determinants of orders  $n-1$  and  $n-2$ . Developing  $d_n$  by means of the last column and row, we obtain  $d_n = 4d_{n-1} - d_{n-2}$ . Hence, by the usual method for such difference equations,

$$d_n = A(2 + \sqrt{3})^n + B(2 - \sqrt{3})^n.$$

But  $d_1 = 2$  and  $d_2 = 7$ . Hence  $A = B = \frac{1}{2}$ , and  $d_n$  is given.

Again,

$$d_n' = \frac{g}{2} \begin{vmatrix} 3 & 3 & 3 & 3 & \cdot & 3 \\ 12 & 11 & 12 & 12 & \cdot & 12 \\ 27 & 20 & 26 & 27 & \cdot & 27 \\ 48 & 29 & 41 & 47 & \cdot & 48 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3n^2 & 9n-7 & \cdot & \cdot & \dots & 3n^2-1 \end{vmatrix}$$

Subtracting the first column from each of the others, we have

$$d_n' = \frac{g}{2} \begin{vmatrix} 3 & 0 & 0 & 0 & \cdot & 0 \\ 12 & -1 & 0 & 0 & \cdot & 0 \\ 27 & -7 & -1 & 0 & \cdot & 0 \\ 48 & -19 & -7 & -1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3n^2 & \cdot & \cdot & \cdot & \cdot & -1 \end{vmatrix} = (-1)^{n-1} \frac{3g}{2}.$$

Hence,

$$x_n = \frac{(-1)^{n-1} 3g}{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}.$$

**467. Proposed by IRA M. DE LONG, University of Colorado.**

Determine the function  $f$ , from the functional relation  $f(x+y) = f(x) + f(y) + 2xy$ .

## I. SOLUTION BY A. COHEN, Johns Hopkins University.

From  $f(x + y) = f(x) + f(y) + 2xy$  we have, when  $y = 0$ ,  $f(x) = f(x) + f(0)$ ; hence,

$$(I) \quad f(0) = 0.$$

Similarly, letting  $y = x, 2x, 3x$  in turn, we have respectively,

$$f(2x) = 2f(x) + 2x^2; \quad f(3x) = 3f(x) + 6x^2; \quad f(4x) = 4f(x) + 12x^2.$$

This suggests the law,

$$(II) \quad f(nx) = nf(x) + n(n-1)x^2.$$

Assuming this and letting  $y = nx$ , we have  $f[(n+1)x] = (n+1)f(x) + n(n+1)x^2$

Thus the law is established for  $n$  a positive integer.

That it also holds for  $m/n$ , a rational fraction, may be seen as follows:

Letting  $y = -x$  and using (I)

$$(III) \quad f(-x) = -f(x) + 2x^2.$$

From this readily follows

$$(III') \quad f(-nx) = -nf(x) + n(n+1)x^2 = -nf(x) + (-n)(-n-1)x^2.$$

$$\text{If} \quad y = -\frac{(n-1)}{n}x, \quad f\left(\frac{x}{n}\right) = f(x) + f\left(-\frac{(n-1)}{n}x\right) - \frac{2(n-1)}{n}x^2.$$

$$\text{Using (III'),} \quad f\left(-\frac{(n-1)}{n}x\right) = (1-n)f\left(\frac{x}{n}\right) + \frac{n(n-1)}{n^2}x^2.$$

$$\text{Hence} \quad f\left(\frac{x}{n}\right) = f(x) + (1-n)f\left(\frac{x}{n}\right) + \frac{1-n}{n}x^2.$$

$$\text{From this follows at once} \quad f\left(\frac{x}{n}\right) = \frac{1}{n}f(x) + \frac{1}{n}\left(\frac{1}{n}-1\right)x^2.$$

Using (II) and then (IV),

$$\begin{aligned} f\left(\frac{m}{n}x\right) &= mf\left(\frac{x}{n}\right) + \frac{m(m-1)}{n^2}x^2 \\ &= \frac{m}{n}f(x) + \frac{m}{n}\frac{1-n}{n}x^2 + \frac{m}{n}\frac{m-1}{n}x^2 = \frac{m}{n}f(x) + \frac{m}{n}\left(\frac{m}{n}-1\right)x^2 \end{aligned}$$

which proves the law (II) for  $n$  any positive rational fraction.

Making use of (III'), the law is readily seen to hold for  $n$  a negative rational fraction.

Assuming  $f(x)$  to be a continuous function of  $x$ , the law will hold for  $n$  any real number, as can be shown by using a sequence of rational numbers whose limit is  $n$ .

Once this is established, the rest follows immediately.

Assuming that  $f(x)$  has a derivative, we have on differentiating (II) with respect to  $n$ ,  $x \frac{df(nx)}{d(nx)} = f(x) + (2n-1)x^2$ , which holds for all real values of  $n$ . In particular, for  $n = 1$ ,

$x \frac{df(x)}{dx} = f(x) + x^2$ , or  $xd f - f dx = x^2 dx$ . An obvious integrating factor is  $1/x^2$ . Introducing this and integrating we have  $f/x = x + c$ ; whence

$$(V) \quad f(x) = x^2 + cx, \text{ where } c \text{ is any constant.}$$

## II. SOLUTION BY NATHAN ALTSCHILLER, University of Oklahoma.

Let us determine  $f$  from the relation

$$(f) \quad f(x+y) = f(x) + f(y) + 2cxy,$$

where  $c$  is a given constant.

If in (f) we put  $y = x$ , we have

$$(1) \quad f(2x) = 2f(x) + 2cx^2$$

and if we put  $y = 2x$ , we have

$$f(x+2x) = f(x) + f(2x) + 2c \cdot x \cdot 2x,$$

which may be written, taking (1) into account,

$$(2) \quad f(3x) = 3f(x) + 3 \cdot 2cx^2.$$

In order to prove that this last formula is general, i. e., that we have, for any positive integral value of  $n$ ,

$$(F) \quad f(nx) = nf(x) + n(n-1)cx^2,$$

we assume that we have for  $n = m$

$$(3) \quad f(mx) = mf(x) + m(m-1)cx^2$$

and shall prove that (F) holds then for  $n = m + 1$ . On the strength of (f) we have

$$f[(m+1)x] = f(mx) + f(x) + 2cmx^2;$$

whence, substituting for  $f(mx)$  its development from (3) and simplifying,

$$f[(m+1)x] = (m+1)f(x) + (m+1)m cx^2.$$

If in (f) we put  $x = y = 0$ , we have

$$f(0+0) = f(0) + f(0) - 2c \cdot 0 \cdot 0$$

or  $f(0) = 2f(0)$ , hence  $f(0) = 0$ . Now for  $y = -x$ , (f) gives

$$f(x-x) = f(x) + f(-x) - 2cx^2$$

and since  $f(0) = 0$ , we have, solving for  $f(-x)$ ,

$$(4) \quad f(-x) = -f(x) + 2cx^2.$$

This formula shows that (F) holds for  $n = -1$ . Hence in order to prove that (F) is valid for all negative integral values of  $n$ , it is sufficient to prove that it holds for  $n = -(m+1)$ , if it holds for  $n = -m$ . We assume therefore that

$$(5) \quad f(-mx) = -mf(x) + (-m)(-m-1)cx^2.$$

We have from (f)

$$f[-(m+1)x] = f(-mx) + f(-x) + 2c(-mx)(-x).$$

Replacing  $f(-mx)$  and  $f(-x)$  by their developments from (5) and (4), respectively, and simplifying, we obtain

$$f[-(m+1)x] = -(m+1)f(x) + [-(m+1)][-(m+1)-1]cx^2.$$

Since (F) is valid for all integral values of  $n$ , positive or negative, we have,  $k$  being any integer,

$$(6) \quad f(x) = f\left(k \cdot \frac{x}{k}\right) = kf\left(\frac{x}{k}\right) + k(k-1)c\left(\frac{x}{k}\right)^2,$$

which, when solved for  $f(x/k)$ , may be written in the form

$$(7) \quad f\left(\frac{1}{k} \cdot x\right) = \frac{1}{k}fx + \frac{1}{k}\left(\frac{1}{k} - 1\right)cx^2.$$

Hence, (F) is valid for  $n = 1/k$ . Now let  $p, q$  be any two integers, positive or negative. We may write, since  $p$  is an integer,

$$f\left(p \cdot \frac{x}{q}\right) = pf\left(\frac{x}{q}\right) + p(p-1)c\frac{x^2}{q^2}$$

or, taking (7) into consideration and simplifying,

$$f\left(\frac{p}{q} \cdot x\right) = \frac{p}{q}fx + \frac{p}{q}\left(\frac{p}{q} - 1\right)cx^2.$$

We have thus proved that (F) is valid for any rational value of  $n$ . Any irrational quantity  $s$  may be considered as the limit of a variable rational quantity, and since (F) holds for all these rational values, it also holds for  $s$ , if the function  $f$  is continuous, which we suppose. Hence, (F)

is valid for any real value of  $n$ . Since in the above considerations no restrictions were put upon  $x$ , ( $F$ ) holds for any value of  $x$ , real or complex, if we assume, as we may, that ( $f$ ) holds for such values of the variables. Thus we may write:

$$(8) \quad f(z) = f(u + vi) = f(u) + f(vi) + 2cuv i.$$

But

$$f(u) = f(u \cdot 1) = uf(1) + u(u-1)c(1)^2,$$

$$f(vi) = f(v \cdot i) = vf(i) + v(v-1)c(i)^2;$$

hence, (8) may be written:

$$(9) \quad f(z) = f(u + vi) = uf(1) + vf(i) + c(u + vi)^2 - (u - v)c.$$

In order to determine the value of  $f(1)$  we put  $u = 1$ , and  $v = 0$  and we have from (9)  $f(1) = f(1)$ , which shows that the value of  $f(1)$  is arbitrary. By putting  $u = 0$ ,  $v = 1$ , we may show in the same way that  $f(i)$  is arbitrary. Hence denoting  $f(1) - c$  by  $\alpha$  and  $f(i) + c$  by  $\beta$ , where  $\alpha$  and  $\beta$  are two arbitrary quantities, (9) may be written

$$f(u + vi) = c(u + vi)^2 + \alpha u + \beta v.$$

If  $\alpha = \beta = 0$ , i. e., if  $f(1) = 1$  and  $f(i) = -1$ , and  $c = 1$  we have

$$f(u + vi) = (u + vi)^2.$$

Also variously solved by T. M. SIMPSON, O. S. ADAMS, ELIJAH SWIFT, E. R. SMITH, HORACE OLSON, A. A. BENNETT, C. F. GUMMER, and J. L. WALSH.

**468. Proposed by H. C. FEEMSTER, York College, Neb.**

In each of the following series find the  $n$ th term and sum:

$$(a) \quad 2 + 5 + 9 + 15 + 24 + \cdots,$$

$$(b) \quad 1 + 6 + 10 + 20 + 35 + \cdots,$$

$$(c) \quad 1 + 5 + 15 + 35 + 70 + \cdots,$$

SOLUTION BY J. L. RILEY, Tahlequah, Okla.

(a) Using the method of differences we have

$$\begin{array}{ccccccc} 3 & 4 & 6 & 9 & \cdots \\ & 1 & 2 & 3 & \cdots \\ & & 1 & 1 & \cdots \\ & & & 0 & \cdots \end{array}$$

$$\begin{aligned} U_n &= 2 + 3(n-1) + \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)(n-3)}{3} \\ &= \frac{n^3 - 3n^2 + 20n - 6}{6}, \text{ the } n\text{th term,} \end{aligned}$$

$$\begin{aligned} S_n &= 2n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)(n-2)(n-3)}{4} \\ &= \frac{n}{24} (n^3 - 2n^2 + 35n + 14), \text{ the sum.} \end{aligned}$$

(b) In the series  $1 + 6 + 10 + 20 + 35 + \cdots$ , let  $U_n = A + Bn + Cn^2 + Dn^3 + En^4$ . Then

$$\begin{cases} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 6, \\ A + 3B + 9C + 27D + 81E = 10, \\ A + 4B + 16C + 64D + 256E = 20, \\ A + 5B + 25C + 125D + 625E = 35. \end{cases}$$